Coherent state of a weakly interacting ultracold Fermi gas

Arnab Ghosh, Sudarson Sekhar Sinha and Deb Shankar Ray [*]

Indian Association for the Cultivation of Science, Jadavpur, Kolkata 700 032, India.

Abstract

We examine the weakly interacting atoms in an ultracold Fermi gas leading to a state of macroscopic coherence, from a theoretical perspective. It has been shown that this state can be described as a fermionic coherent state. These coherent states are the eigenstates of fermionic annihilation operators, the eigenvalues being anti-commuting numbers or Grassmann numbers. By exploiting the simple rules of Grassmann algebra and a close kinship between relations evaluated for more familiar bosonic fields and those for fermionic fields, we derive the thermodynamic limit, the spontaneous symmetry breaking and the quasi-particle spectrum of the fermionic system.

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[*] Email Address: pcdsr@iacs.res.in

I.   INTRODUCTION

Dilute, ultracold, trapped atomic gases have served as model systems for exploring quantum phenomena at the fundamental level [1–4]. One of the basic issues is the condensation of a Fermi gas. Important experimental advances on alkali atoms ^{6}Li and ^{40}K [5, 6] have been made in this regard to study the interaction between the fermions. Since the interaction between cold atoms characterized by s-wave scattering length can be controlled by magnetic field Feshbach resonances, it has been possible to realize both the repulsive (BEC) and the attractive (BCS) regimes and their crossover. In spite of a very large number of theoretical papers published over a decade, a complete analytic solution of the many-body problem along the BEC-BCS crossover still remains illusive [7–9]. However, the use of mean field theory, pseudopotential and consideration of fluctuations around mean field has led interesting advancement in understanding the basic physics of ultracold atomic Fermi gases. For details we refer to the topical review by Giorgini, Pitaevskii and Stringari [9].

To capture the essential physics around the crossover it is necessary to comprehend the formation of Cooper pairs or molecule or BEC condensate or condensation of fermionic atom pairs [10–14]. A close look into this aspect reveals that the description of "Fermi condensate" around this crossover is still somewhat incomplete [14]. This is because of the fact that a simple two-body physics of the resonance on the attractive side of interaction does not support a weakly bound molecular state – a point emphasized by Jin *et. al.* [12]. This implies that many-body effects must prevail in characterizing such a state. In the present study we focus on the description of this state in a weakly interacting Fermi gas. Pauli principle rules out the possibility of macroscopic occupation of a single quantum energy state of fermions. A question, however, remains, whether it is possible to realize a state of macroscopic coherence among them. This is based on exploitation, following Cahill and Glauber [15], of the close parallels between the families of the quasi-probability phase space density functions for bosonic and fermionic fields. A key element of the formulation is the fermionic coherent state defined as a displaced state where the displacement operator acts on the vacuum state. The transformation with the displacement operator displaces a fermionic field operator over an anti-commuting number or Grassmann number [15–18]. Fermionic coherent state is an eigenstate of annihilation operator, eigenvalues being the Grassmann numbers. This is reminiscent of the harmonic oscillator coherent state for which the dis-

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placement operator displaces the bosonic field operator over a classical commuting number [19–21]. Our method for description of the macroscopic state of weakly interacting fermions is based on the fermionic coherent state and the associated algebra of the anti-commuting numbers, which have no clasical analog [15–18]. They form the basis for demonstration of the thermodynamic limit, spontaneous symmetry breaking and the quasi-particle spectrum of the atoms comprising the coherent state. A major goal of the approach is to show that it is possible to achieve a coherent state of weakly interacting fermions for which the scattering length can be either positive or negative. And the state can exist as independent entity irrespective of the details of Cooper pairing or molecule formation.

The paper is organized as follows: In Sec.II we briefly introduce the basic model for weakly interacting many body systems under ultracold condition. Since Grassmann numbers play a crucial role in the formulation of the problem we briefly review in Sec.III the relevant aspects of Grassmann algebra before introducing the fermionic coherent state following Cahill and Glauber [15]. That the state of macroscopic coherence can be realized as a fermionic coherent state is the main theme of this section. We further examine the aspect of thermodynamic limit and spontaneous symmetry breaking . In Sec.IV we derive the quasi-particle spectrum using Bogoliubov transformation. The paper is concluded in Sec.V.

II.   WEAKLY INTERACTING FERMI GAS

To begin with we consider a dilute gas of fermionic atoms. The interaction between them is as usual nonzero only when the range of inter-particle interaction r_0 is much smaller than the average inter-particle distance d, i.e., $r_0 << d = \rho^{-1/3} = (\frac{N}{V})^{1/3}$. This also ensures that the properties of the system can be expressed in terms of a single parameter, scattering length, a. The diluteness criterion in terms s-wave scattering length can be expressed as $|a|\rho^{1/3} << 1$. We also assume that the temperature of the dilute gas is so low that the momentum q as distributed thermally is much smaller than the characteristic momentum $q_c = \hbar/r_0$. The scattering amplitude is independent of momentum q when $q << q_c$. While the temperature for onset of quantum degeneracy goes as $\rho^{2/3}$, temperature for BEC-BCS crossover is of the order of Fermi temperature for a range of dimensionless $k_{\mathbf{F}}|a|$ values, where $k_{\mathbf{F}}$ corresponds to Fermi wave vector [7–9].

The Hamiltonian of the system can be expressed in terms of the field operator $\hat{\psi}(r)$ as

$$\hat{\mathbf{H}} = \int \left(\frac{\hbar^2}{2m} \nabla \hat{\psi}^\dagger(r) \nabla \hat{\psi}(r) \right) dr + \frac{1}{2} \int \hat{\psi}^\dagger(r) \hat{\psi}^\dagger(r') V(r - r') \hat{\psi}(r) \hat{\psi}(r') dr\, dr' \qquad (2.1)$$

where $V(r - r')$ is the two-body potential. For the moment we have not included the external fields. In absence of external fields, the N particles move only as a result of their mutual interactions. The field operator $\hat{\psi}(r)$ annihilates a particle at a position r and can be expressed as $\hat{\psi}(r) = \sum_i \phi_i(r) \hat{a}_i$. $\hat{a}_i(\hat{a}_i^\dagger)$ is the annihilation (creation) operator of a particle in the single particle state $\phi_i(r)$ and these operators obey anti-commutation relations

$$\{\hat{a}_i, \hat{a}_j^\dagger\} = \delta_{ij}; \quad \{\hat{a}_i, \hat{a}_j\} = \{\hat{a}_i^\dagger, \hat{a}_j^\dagger\} = 0 \qquad (2.2)$$

The wave functions $\phi_i(r)$ satisfy orthonormal condition

$$\int \phi_i^*(r) \phi_j(r) = \delta_{ij} \qquad (2.3)$$

The field operator then follows anti-commutation relation $\{\hat{\psi}(r), \hat{\psi}^\dagger(r')\} = \sum_i \phi_i(r) \phi_i^*(r')$ $= \delta(r - r')$. Now expressing the field operator $\hat{\psi}(r)$ and the interaction term, $V(r, r')$ ($\equiv V(r - r')$) as $\hat{\psi}(r) = \sum_k \hat{a}_k \frac{e^{ik.r}}{\sqrt{V}}$ and $V_q = \int V(r) e^{-\frac{i.q.r}{\hbar}} dr$, we may simplify the Hamiltonian as

$$\hat{\mathbf{H}} = \sum_k \frac{\hbar^2 k^2}{2m} \hat{a}_k^\dagger \hat{a}_k + \frac{1}{2V} \sum_{k_1, k_2, q} V_q \hat{a}_{k_1+q}^\dagger \hat{a}_{k_2-q}^\dagger \hat{a}_{k_2} \hat{a}_{k_1} \qquad (2.4)$$

We take into account of the momentum conservation during a two-fermion interaction such that every single term within the summation describes the annihilation of a pair of fermionic atoms and the creation of another pair. Since the description of the macroscopic properties of the gas does not require the detailed form of the two-body interaction, microscopic potential $V(r)$ may be replaced by a soft, effective potential $V_{eff}(q)$ which, in turn, gives the low momentum value of its Fourier transform V_0. For such small momenta $q << q_c = \hbar/r_0$ we are allowed to consider V_0 or the s-wave scattering length 'a' becomes independent of its momentum q [7–9].

The Hamiltonian therefore reduces to

$$\hat{\mathbf{H}} = \sum_k \frac{\hbar^2 k^2}{2m} \hat{a}_k^\dagger \hat{a}_k + \frac{V_0}{2V} \sum_{k_1, k_2, q} \hat{a}_{k_1+q}^\dagger \hat{a}_{k_2-q}^\dagger \hat{a}_{k_2} \hat{a}_{k_1} \qquad (2.5)$$

This is the starting Hamiltonian for weakly interacting Fermi atoms and henceforth will be used in the following sections. Finally we emphasize that the Hamiltonian considered here

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(Eqs. 2.1-2.5) follows Greiner [21] and does not refer to specific spin states. This is an important point of departure from many theoretical approaches to quantum Fermi liquid [7] and BCS-BEC crossover [7–9].

To proceed further for a systematic description we first separate out the field operator $\hat{\psi}(r)$ into the coherent term ($i = \mathbf{F}$) and the incoherent term ($i \neq \mathbf{F}$) as;

$$\hat{\psi}(r) = \phi_{\mathbf{F}}(r)\hat{a}_{\mathbf{F}} + \sum_{i \neq \mathbf{F}} \phi_i(r)\hat{a}_i \tag{2.6}$$

We do this separation by taking a hint from Bogoliubov approximation [22] for bosons and the close parallelism between the expressions developed for bosonic and fermionic fields by Cahill and Glauber [15]. We may note that an implementation of Bogoliubov approximation [22] for fermionic fields requires replacement of $\hat{a}_{\mathbf{F}}$ and $\hat{a}_{\mathbf{F}}^{\dagger}$ by anti-commuting numbers, say $y_{\mathbf{F}}$ and $y_{\mathbf{F}}^*$ [15–18]. One can always multiply these numbers by a numerical phase factor as $y_{\mathbf{F}}e^{i\theta}$ and $y_{\mathbf{F}}^*e^{-i\theta}$ without changing any physical property. This phase 'θ' plays, as we will see, a major role in characterizing the coherence in the weakly interacting ultracold Fermi gas. This reflects the gauge symmetry exhibited by all the physical equations of the problem. Making an explicit choice for the value of the phase actually corresponds to a formal breaking of gauge symmetry.

Equivalently the Bogoliubov approximation for fermions is equivalent to treating the macroscopic component $\phi_{\mathbf{F}}(r)\hat{a}_{\mathbf{F}}$ as a "classical" non-commuting field $\psi_{\mathbf{F}}(r)$ so that $\hat{\psi}(r)$ may be rewritten as

$$\hat{\psi}(r) = \psi_{\mathbf{F}}(r) + \delta\hat{\psi}(r) \tag{2.7}$$

The above ansatz for the fermionic field operator can be interpreted as the expectation value $\langle\hat{\psi}(r)\rangle$ different from zero. This is not possible if the coherent state is a particle number eigenstate. Although such coherent states are well-known for bosonic fields and have been the basis for understanding BEC [23–27], an extension of the scheme to their fermionic counterpart is not straight-forward. The primary reason may be traced to a basic issue. Since fermions anti-commute their eigenvalues must be anti-commuting numbers, as pointed out by Schwinger [16]. Such numbers are Grassmann numbers which can be dealt with by simple techniques of Grassmann algebra [17, 18]. In what follows in the next section we make a little digression on this algebra centering around fermionic coherent state following Cahill and Glauber [15] which forms the integral part of the description of the

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interacting fermions near the crossover.

Although an important guideline of the present formulation is the close parallels between some aspects of bosons and fermions it is important to highlight the other essentials. The following digression may be interesting. While for a bosonic gas the condensed phase is formed by the particles in the zero energy ground state, in the fermionic gas the particles may "condense" [28] in the Fermi level. Such a prediction was made many decades ago by Kothari and Nath [28] in course of examination of Born's reciprocity principle [29]. In the present scenario, we note that the BEC-BCS crossover temperature for a Fermi gas is less than or equal to the Fermi temperature [10–14], a point that hints towards this assertion. In what follows we will show that a macroscopic state of weakly interacting ultracold fermions can give rise to a fermionic coherent state.

III. GRASSMANN ALGEBRA AND FERMIONIC COHERENT STATES; A CONNECTION TO MACROSCOPIC COHERENCE OF FERMIONS:

A. Grassmann variables and their properties:

We now summarize some of the properties of the anti-commuting classical variables relevant for our future discussions. These variables can be treated within the scope of Grassmann algebra, which are well studied in mathematics and field theory [17]. They possess very uncommon properties.

Let, $\mathbf{y} = \{y_i\}, i = 1, 2, ...n.$ define a set of generators which satisfy anti-commutation properties,

$$y_i y_j + y_j y_i \equiv \{y_i, y_j\} = 0 \qquad (3.1)$$

This, in particular, implies that $y_i^2 = 0$, for any given 'i'. Since the square of every Grassmann monomial vanishes, a nonzero Grassmann monomial can not be an ordinary real, imaginary, or complex number. In other words, Grassmann variables are nilpotent, an important property for the treatment of fermions [18]. The anti-commuting numbers y_i and their complex conjugates y_i^* are independent numbers and satisfy

$$\{y_i, y_i^*\} = 0 \qquad (3.2)$$

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They also anti-commute with their fermionic operators

$$\{y_i, \hat{a}_j\} = 0 ; \qquad \{y_i, \hat{a}_j^\dagger\} = 0 \tag{3.3}$$

And hermitian conjugation reverses the order of all fermionic quantities, both the operators and the anti-commuting numbers. For instance, we have

$$(\hat{a}_1 y_2 \hat{a}_3^\dagger y_4^*)^\dagger = y_4 \hat{a}_3 y_2^* \hat{a}_1^\dagger \tag{3.4}$$

An analytic function of only one Grassmann variable can be expressed as a simple Taylor expansion

$$f(y) = f_0 + f_1 y \qquad (\text{since } y^2 = 0) \tag{3.5}$$

Thus exponential function for Grassmann variables has only two terms

$$e^y = \sum_{k=0}^{\infty} \frac{y^k}{k!} = 1 + y \quad (\text{since } y^2 = 0) \tag{3.6}$$

Therefore for a function of single Grassmann variable, integration (integration due to Berezin [17]) is identical to differentiation. The fundamental rules of integration over the complex Grassmann variables are as follows;

$$\int dy_i = 0 \qquad \int dy_i^* = 0 \tag{3.7}$$

$$\int dy_i \, y_j = \delta_{ij} \qquad \int dy_i^* \, y_j^* = \delta_{ij} \tag{3.8}$$

This difference between ordinary variables and Grassmann variables has many interesting consequences. We are typically concerned with pairs of anti-commuting variables y_i and y_i^*, and for such pairs we will confine ourselves to the notation

$$\int d^2 y_i = \int dy_i^* dy_i \tag{3.9}$$

in which the differential of the conjugated variable dy_i^* comes first and we keep in mind that

$$dy_i^* dy_i = -dy_i dy_i^* \tag{3.10}$$

We can write the multiple integrals over such sets as

$$\int d^2 \mathbf{y} \equiv \int \prod_i dy_i^* dy_i \tag{3.11}$$

B.   Fermionic coherent states:

We are now in a position to introduce the fermionic coherent states. In analogy to harmonic oscillator coherent state $|\boldsymbol{\alpha}\rangle$ [19] defined as a displaced state where the displacement operator $\hat{\mathsf{D}}(\boldsymbol{\alpha}) = \exp\left(\sum_i (\alpha_i \hat{a}_i^\dagger - \alpha_i^* \hat{a}_i)\right)$ acts on the vacuum $|0\rangle$ as $|\boldsymbol{\alpha}\rangle = \hat{\mathsf{D}}(\boldsymbol{\alpha})|0\rangle$, $\{\alpha_i\}$ being a set of complex numbers [19–21], it is possible to construct a displacement operator for fermions [15] as

$$\hat{\mathbf{D}}(\mathbf{y}) = \exp\left(\sum_i (\hat{a}_i^\dagger y_i - y_i^* \hat{a}_i)\right) \tag{3.12}$$

for a set of $\mathbf{y} = \{y_i\}$ Grassmann variables and the fermionic coherent state can then be constructed by the action of this displacement operator (Eq. 3.12) on the vacuum state as

$$|\mathbf{y}\rangle = \hat{\mathbf{D}}(\mathbf{y})|0\rangle \tag{3.13}$$

1.   Displacement operator and its properties:

An important property of the Grassmann variables is that on multiplication with fermionic annihilation or creation operators, their anti-commutivity cancels that of the operators. Thus the operators $\hat{a}_i^\dagger y_i$ and $y_j^* \hat{a}_j$ simply commute for $i \neq j$ [15]. Therefore the displacement operator may be written as the product

$$\begin{aligned}
\hat{\mathbf{D}}(\mathbf{y}) &= \prod_i \exp\left(\hat{a}_i^\dagger y_i - y_i^* \hat{a}_i\right) \\
&= \prod_i \left[1 + \hat{a}_i^\dagger y_i - y_i^* \hat{a}_i + \left(\hat{a}_i^\dagger \hat{a}_i - \frac{1}{2}\right) y_i^* y_i\right]
\end{aligned} \tag{3.14}$$

Similarly the annihilation operators $\hat{a}_k$ commutes with all the operators $\hat{a}_i^\dagger y_i$ and $y_i^* \hat{a}_i$ for $k \neq i$, and so we may compute the displaced annihilation operator by ignoring all modes except the k-th one

$$\begin{aligned}
\hat{\mathbf{D}}^\dagger(\mathbf{y})\hat{a}_k \hat{\mathbf{D}}(\mathbf{y}) &= \prod_i \exp\left(y_i^* \hat{a}_i - \hat{a}_i^\dagger y_i\right).\hat{a}_k.\prod_j \exp\left(\hat{a}_j^\dagger y_j - y_j^* \hat{a}_j\right) \\
&= \exp\left(y_k^* \hat{a}_k - \hat{a}_k^\dagger y_k\right).\hat{a}_k.\exp\left(\hat{a}_k^\dagger y_k - y_k^* \hat{a}_k\right) \\
&= \hat{a}_k + y_k
\end{aligned} \tag{3.15}$$

Similarly we have

$$\hat{\mathbf{D}}^\dagger(\mathbf{y})\hat{a}_k^\dagger \hat{\mathbf{D}}(\mathbf{y}) = \hat{a}_k^\dagger + y_k^* \tag{3.16}$$

By using the displacement relation [Eq (3.15)] we may show that the coherent state is an eigenstate of every annihilation operator $\hat{a}_k$;

$$
\begin{aligned}
\hat{a}_k|\mathbf{y}\rangle = \hat{a}_k\hat{\mathbf{D}}(\mathbf{y})|0\rangle &= \hat{\mathbf{D}}(\mathbf{y})\hat{\mathbf{D}}^\dagger(\mathbf{y})\hat{a}_k\hat{\mathbf{D}}(\mathbf{y})|0\rangle \\
&= \hat{\mathbf{D}}(\mathbf{y})(\hat{a}_k + y_k)|0\rangle \\
&= y_k\hat{\mathbf{D}}(\mathbf{y})|0\rangle = y_k|\mathbf{y}\rangle
\end{aligned}
\tag{3.17}
$$

The adjoint of the coherent state $|\mathbf{y}\rangle$ can be similarly defined as $\langle\mathbf{y}|\hat{a}_k^\dagger = \langle\mathbf{y}|y_k^*$.
The inner product of two coherent states is

$$
\langle\mathbf{y}'|\mathbf{y}\rangle = \exp\left(\sum_i y_i'^* y_i - \frac{1}{2}(y_i'^* y_i' + y_i^* y_i)\right)
\tag{3.18}
$$

and using the completeness properties of the coherent states, any arbitrary coherent state $|\mathbf{y}'\rangle$ can be expanded [15, 18] as

$$
|\mathbf{y}'\rangle = \int d^2\mathbf{y}\langle\mathbf{y}|\mathbf{y}'\rangle|\mathbf{y}\rangle
\tag{3.19}
$$

which immediately follows from the resolution of identity

$$
\int d^2\mathbf{y}|\mathbf{y}\rangle\langle\mathbf{y}| = \mathbf{I}
\tag{3.20}
$$

C. Weakly interacting Fermi gas in coherent state representation:

The key idea behind the present formulation is whether a state of macroscopic coherence of an weakly interacting Fermi gas can be achieved as a fermionic coherent state. Such an approach gives the result which is equivalent to that of the problem of fixed number of particles N in the limit $N \longrightarrow \infty$ for the BEC case [23–27]. In the same spirit, we can extend the coherent state approach to its fermionic counterpart. Before proceeding further, we discuss the physical states and operators, which will be helpful for the future development.

1. *Physical states and operators :*

According to Cahill and Glauber, [15] a state $|\psi\rangle$ is physical if it changes at most by a phase when subjected to a rotation of angle 2π about any axis,

$$\mathbf{U}(\hat{n}, 2\pi)|\psi\rangle = e^{i\theta}|\psi\rangle \tag{3.21}$$

Since, fermions carry half-integer spin, a state of any odd number of fermions changes by the phase factor -1, while even number of fermions are invariant under such 2π rotation. Thus physical states are linear combinations of states with even number of fermions or linear combination of odd number of fermions. On the other hand, a linear combination of states $|0\rangle$ and $|1\rangle$ like $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ must be unphysical since it changes to a different state under 2π rotation.

An operator is physical if it maps physical states onto physical states. Physical operators are either even or odd. In all physical contexts, if we consider $\hat{N} = \sum_k \hat{a}_k^\dagger \hat{a}_k$ as the total fermion number which is conserved, then any state arising from the eigenstate of $\hat{N}$ must remain an eigenstate of $\hat{N}$. This can be derived on the basis of $\mathbf{U}(1)$ invariance of all the interactions under the transformation $\hat{U}(\theta) = e^{i\theta\hat{N}}$. We have the transformation on $\hat{a}_i$ and $\hat{a}_i^\dagger$ as follows;

$$e^{-i\theta\hat{N}}\hat{a}_i e^{i\theta\hat{N}} = e^{i\theta}\hat{a}_i \tag{3.22}$$

and

$$e^{-i\theta\hat{N}}\hat{a}_i^\dagger e^{i\theta\hat{N}} = e^{-i\theta}\hat{a}_i^\dagger \tag{3.23}$$

for fermion conserving interactions that involve $\hat{a}_k$ and $\hat{a}_k^\dagger$; and the phase factors get cancel out. We now emphasize that the coherent states undergo a simple change under this transformation

$$\hat{U}(\theta)|\mathbf{y}\rangle = e^{i\theta\hat{N}}|\mathbf{y}\rangle = |e^{i\theta}\mathbf{y}\rangle, \tag{3.24}$$

However the scalar product remains invariant

$$\langle e^{i\theta}\mathbf{y}|e^{i\theta}\mathbf{y}\rangle = \langle\mathbf{y}|\mathbf{y}\rangle \tag{3.25}$$

2. Spontaneous symmetry breaking:

We now note that one can always multiply the coherent state by an arbitrary phase factor $e^{i\theta}$ without changing any physical property (as far as physical states are concerned). This is the manifestation of gauge symmetry in the problem. Physically, the lack of a force responsible for phase stabilization of the system is the origin for the random phase of a condensate. However, when one refers to Fermi systems the macroscopic system is expected to choose spontaneously a particular phase "θ". Making an explicit choice for the phase "θ" in spite of the lack of a preferred phase value (referred to as a spontaneous breaking of gauge symmetry) implies that the macroscopic state is in or close to a coherent state. From the symmetry point of view, the situation is quite interesting and can be further elaborated as follows.

The states $|\mathbf{y}\rangle$ are not invariant under the number operator $\hat{N} = \sum_k \hat{a}_k^\dagger \hat{a}_k$, while the Hamiltonian (Eq. 2.5) commutes with $\hat{N}$, i.e.,

$$e^{i\theta\hat{N}}|\mathbf{y}\rangle = |e^{i\theta}\mathbf{y}\rangle; \qquad e^{i\theta\hat{N}}\hat{\mathbf{H}}e^{-i\theta\hat{N}} = \hat{\mathbf{H}} \tag{3.26}$$

The operator $e^{i\theta\hat{N}}$ applied to $|\mathbf{y}\rangle$ produces a state with the same energy, with a phase shifted by "θ". Since the overlap of coherent states obey $\langle\mathbf{y}'|\mathbf{y}\rangle = \exp\left(\sum_i y_i'^* y_i - \frac{1}{2}(y_i'^* y_i' + y_i^* y_i)\right)$ [Eq. (3.18)], any two different states $|\mathbf{y}\rangle$, $|\mathbf{y}'\rangle$ with different phase factors, $|\mathbf{y}\rangle$, $|e^{i\theta}\mathbf{y}\rangle$, parameterized by a phase variable $0 < \theta < 2\pi$, are macroscopically distinct. While the microscopic Hamiltonian ($\hat{\mathbf{H}}$) [Eq. 2.5] has global $\mathbf{U}(1)$ symmetry, the state does not possess such symmetry, since adding a phase factor to the state $|\mathbf{y}\rangle$ produces a different state altogether.

3. Population fluctuation and phase stabilization:

One can easily calculate the population fluctuation in the coherent state given by Eq. (3.13) as

$$\langle\Delta\hat{N}^2\rangle = \langle\hat{N}^2\rangle - \langle\hat{N}\rangle^2 = \sum_k y_k^* y_k = \langle\hat{N}\rangle \tag{3.27}$$

Here we have to use the property [15] of the fermionic number operator $\hat{N} = \sum_k \hat{a}_k^\dagger \hat{a}_k$ and the nilpotency of the Grassmann variables i.e., $(y_k^* y_k)^2 = 0$, for each k.

Then the phase fluctuation in the coherent state is given by

$$\langle \Delta \hat{\theta}^2 \rangle = \frac{1}{4 \langle \Delta \hat{N}^2 \rangle} = \frac{1}{4 \langle \hat{N} \rangle} = \frac{1}{4 \sum_k y_k^* y_k} \tag{3.28}$$

Since $\hat{N}$ and $\hat{\theta}$ are the canonically conjugate pairs, the phase is stabilized $\langle \Delta \hat{\theta}^2 \rangle << 1$, only at the cost of the enhanced fluctuation in population i.e., $\langle \Delta \hat{N}^2 \rangle >> 1$. Since the coherent state can be expanded as a coherent superposition of the particle number eigenstates [Eq. (3.13)], the constructive and destructive interferences among different number eigenstates result in the stabilized phase but with finite particle number noise. Population fluctuation and phase stabilization are the typical signatures of spontaneous symmetry breaking [27].

4. Thermodynamic limit; Grassmann-Bogoliubov approximation:

Once a macroscopic state of interacting fermionic atoms is realized as the coherent state, neither $\hat{a}_{\mathbf{F}}$ nor $\hat{a}_{\mathbf{F}}^\dagger$ annihilates the state. Since, we are interested in the behaviour of a gas of fermionic atoms, i.e., in the large particle number and volume, it is necessary to consider the so-called thermodynamic limit $N \longrightarrow \infty$, $V \longrightarrow \infty$ but with constant density $\rho = \frac{N}{V}$. In this limit, the anti-commutation relation between fermionic operators $\hat{a}_{\mathbf{F}}$ and $\hat{a}_{\mathbf{F}}^\dagger$ becomes

$$\frac{\{\hat{a}_{\mathbf{F}}, \hat{a}_{\mathbf{F}}^\dagger\}}{V} = \frac{\hat{a}_{\mathbf{F}} \hat{a}_{\mathbf{F}}^\dagger + \hat{a}_{\mathbf{F}}^\dagger \hat{a}_{\mathbf{F}}}{V} = \frac{1}{V} \longrightarrow 0 \qquad (\text{when } V \longrightarrow \infty) \tag{3.29}$$

In the limit, $V \longrightarrow \infty$, we are allowed to forget the operator character of $\hat{a}_{\mathbf{F}}$ and $\hat{a}_{\mathbf{F}}^\dagger$ and they can be replaced by numbers, i.e., we obtain the "classical" limit of the fermionic operators. To make this point explicit, let us define

$$\hat{a}_{\mathbf{F}}^\dagger \hat{a}_{\mathbf{F}} = \hat{N}_{\mathbf{F}} \tag{3.30}$$

where $\hat{N}_{\mathbf{F}}$ represents the number operator for the particles in the coherent state. Because of $\langle \mathbf{y} | \hat{N}_{\mathbf{F}} | \mathbf{y} \rangle = y_{\mathbf{F}}^* y_{\mathbf{F}} \neq 0$ and the anti-commuting properties of Grassmann variables (Eq. 3.1), it follows

$$\frac{\langle \mathbf{y} | \{\hat{a}_{\mathbf{F}}, \hat{a}_{\mathbf{F}}^\dagger\} | \mathbf{y} \rangle}{V} = \frac{1 + \{y_{\mathbf{F}}, y_{\mathbf{F}}^*\}}{V} = \frac{1}{V} \longrightarrow 0 \tag{3.31}$$

This leads to the natural starting point of what we may call the Grassmann-Bogoliubov approximation. Here one can identify $y_{\mathbf{F}}^* y_{\mathbf{F}} = N_{\mathbf{F}}(\approx N)$ as the average particle number of the state of macroscopic coherence in the thermodynamic limit.

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One point is to be noted here. One may conclude that anti-commutation obeyed by fermions do not have a classical analogy since they do not fulfill classical Poisson brackets as obeyed by bosonic commutation relations. But this should not lead to misunderstanding. The number operator $\hat{N} = \sum_k \hat{N}_k$ and the Hamiltonian operator $\hat{H} = \sum_k \epsilon_k \hat{N}_k$ have classical limits because they are bilinear in $\hat{a}_k$, $\hat{a}_k^\dagger$. For the particles obeying Fermi-Dirac statistics, only quantities like charge, energy, current density or number density are measurable classically because they are bilinear combination of field amplitudes $\hat{a}_k$, $\hat{a}_k^\dagger$ [18]. The amplitudes of the Fermi field is linear in $\hat{a}_k$, $\hat{a}_k^\dagger$ and can not be measured classically. Extrapolating the idea a bit further we can emphasize that Grassmann fields themselves and fermionic field operators are, by construction, fermionic while a product of even number of fermionic quantities or Grassmann variables is bosonic which makes it experimentally relevant [15, 18, 30].

With this, we now return to our starting Hamiltonian (Eq. 2.5) characterizing the weakly interacting Fermi atoms. In the next section the energy spectrum is calculated order by order under Grassmann-Bogoliubov transformation.

IV. QUASI-PARTICLE SPECTRUM AND QUANTUM FLUCTUATIONS:

A. Lowest-order approximation:

In the first approximation, we can neglect all the terms in the Hamiltonian (Eq. 2.5) containing the operators $\hat{a}_k$ and $\hat{a}_k^\dagger$ with $k \neq \mathbf{F}$. This implies that the quantum fluctuation $(\delta\hat{\psi}(r))$ in Eq. (2.5) can be ignored. Under such approximation the coherent state must be occupied by macroscopically large number of particles i.e. $N_\mathbf{F} \sim N$. The replacement of $\hat{a}_\mathbf{F}$ and $\hat{a}_\mathbf{F}^\dagger$ by '$y_\mathbf{F}e^{i\theta}$' and '$y_\mathbf{F}^*e^{-i\theta}$' respectively, then becomes quite straight forward. This substitution can not be made for a realistic potential since it would result in a poor approximation at short distances of order r_0, where the potential is strong and correlations are important. The replacement is instead accurate in the case of a soft potential whose perturbation is small at all distances [27]. The energy of the interacting system in the lowest order therefore takes the form:

$$E_\mathbf{F} = \frac{\hbar^2 k_\mathbf{F}^2}{2m}\hat{a}_\mathbf{F}^\dagger \hat{a}_\mathbf{F} + \frac{V_0}{2V}\hat{a}_\mathbf{F}^\dagger \hat{a}_\mathbf{F}^\dagger \hat{a}_\mathbf{F} \hat{a}_\mathbf{F} = \frac{\hbar^2 k_\mathbf{F}^2}{2m}y_\mathbf{F}^* y_\mathbf{F} + \frac{V_0}{2V}y_\mathbf{F}^* y_\mathbf{F}^* y_\mathbf{F} y_\mathbf{F} \tag{4.1}$$

To the same order of approximation, one can easily express the parameter V_0 of Eq (4.1) in terms of the scattering length a, using the result $V_0 = \frac{4\pi\hbar^2 a}{m}$, under Born approximation. However, due to the anti-commuting nature of the Grassmann variables [Eq. (3.1)] the energy is given by only the first term in Eq. (4.1), i.e., $E_{\mathbf{F}} = \frac{\hbar^2 k_{\mathbf{F}}^2}{2m} N_{\mathbf{F}} = \mu N_{\mathbf{F}}$, where, $N_{\mathbf{F}} = y_{\mathbf{F}}^* y_{\mathbf{F}}$ represents the average number of particles and 'μ' is the chemical potential of the weakly interacting fremions in the coherent state.

B.   Higher order approximation; calculation of quasi-particle spectrum:

The result $E_{\mathbf{F}} = \mu N_{\mathbf{F}}$ for the energy of the interacting fermions in the lowest order has been obtained by taking into account of Eq (2.5) only for the particle operators $\hat{a}_k$ and $\hat{a}_k^\dagger$ with $k = \mathbf{F}$. The terms containing only one particle operator with $k \neq \mathbf{F}$ do not enter into the Hamiltonian (Eq. 2.5) because of momentum conservation. By retaining all the quadratic terms in the particle operators with $k \neq \mathbf{F}$, in the next higher order, we obtain the following decomposition of the Hamiltonian;

$$\hat{\mathbf{H}} = \sum_k \frac{\hbar^2 k^2}{2m} \hat{a}_k^\dagger \hat{a}_k + \frac{V_0}{2V} \hat{a}_{\mathbf{F}}^\dagger \hat{a}_{\mathbf{F}}^\dagger \hat{a}_{\mathbf{F}} \hat{a}_{\mathbf{F}} + \frac{V_0}{2V} \sum_{k \neq \mathbf{F}} \left(4\hat{a}_{\mathbf{F}}^\dagger \hat{a}_k^\dagger \hat{a}_{\mathbf{F}} \hat{a}_k + \hat{a}_k^\dagger \hat{a}_{-k}^\dagger \hat{a}_{\mathbf{F}} \hat{a}_{\mathbf{F}} + \hat{a}_{\mathbf{F}}^\dagger \hat{a}_{\mathbf{F}}^\dagger \hat{a}_k \hat{a}_{-k} \right) \quad (4.2)$$

Now, the replacement of $\hat{a}_{\mathbf{F}}^\dagger$ and $\hat{a}_{\mathbf{F}}$ by $y_{\mathbf{F}}^* e^{-i\theta}$ and $y_{\mathbf{F}} e^{i\theta}$, as carried out previously, yields the following expression for the Hamiltonian;

$$\hat{\mathbf{H}} = \sum_k \frac{\hbar^2 k^2}{2m} \hat{a}_k^\dagger \hat{a}_k + \frac{V_0}{2V} y_{\mathbf{F}}^* y_{\mathbf{F}}^* y_{\mathbf{F}} y_{\mathbf{F}} + \frac{V_0}{2V} \sum_{k \neq \mathbf{F}} \left(4 y_{\mathbf{F}}^* \hat{a}_k^\dagger y_{\mathbf{F}} \hat{a}_k + \hat{a}_k^\dagger \hat{a}_{-k}^\dagger y_{\mathbf{F}} y_{\mathbf{F}} + y_{\mathbf{F}}^* y_{\mathbf{F}}^* \hat{a}_k \hat{a}_{-k} \right) \quad (4.3)$$

The third term of this Hamiltonian represents the self-energy of the excited states due to the interaction, simultaneous creation of the excited states at momenta k and $-k$ and simultaneous annihilation of the excited states, respectively. But, the simultaneous creation and annihilation of the excited states do not contribute to the Hamiltonian ($\hat{\mathbf{H}}$) due to the Grassmannian nature of the fermionic field variables. Introducing the relevant interaction coupling constant g fixed by the s-wave scattering length a as

$$g = \frac{4\pi\hbar^2 a}{m} \quad (4.4)$$

the Hamiltonian (Eq. 4.3) can be written as

$$\hat{\mathbf{H}} = E_{\mathbf{F}} + \frac{1}{2} \sum_k{}' (E_k + 2\rho g)(\hat{a}_k^\dagger \hat{a}_k + \hat{a}_{-k}^\dagger \hat{a}_{-k}) \quad (4.5)$$

Here $\sum_k{}'$ sign indicates that the terms $k = \mathbf{F}$ are omitted from the summation. ρ represents the fermionic density expressed as $\rho = \frac{y_\mathbf{F}^* y_\mathbf{F}}{V} = \frac{N_\mathbf{F}}{V}(\approx \frac{N}{V})$. The physical coupling constant g renormalizes the effective potential V_0. E_k refers to the energy when the interaction $g = 0$ and is given by $E_k = \frac{\hbar^2 k^2}{2m}$. Unlike the bosonic case, the Hamiltonian in Eq (4.5) is peculiar in a sense that it has no terms with creation operators only or annihilation operators only. In other words, bosonic Hamiltonian does not conserve the number of particles. On the other hand in all physical contexts that have been explored experimentally, the number of fermions or more generally, the number of fermions minus the number of antifermions is strictly conserved [15]. This conservation law leads to further restriction on the permissible states of the field. If a system starts with a state of fixed number of fermions, the conservation law restricts the set of accessible states considerably more than the 2π superselection rule mentioned earlier. Transitions can not be made, for example with different even number of fermions or between states with different odd number of fermions.

We now look for a solution of the problem (Eq. 4.5), i.e., energy eigenvalues of the Hamiltonian. Since the coherent state, represents a combination of unperturbed eigenfunctions, neither $\hat{a}_k$ nor $\hat{a}_k^\dagger$ annihilate this state. The problem can be solved exactly by a canonical transformation, namely Grassmann-Bogoliubov transformation. To this end we introduce new operators

$$
\begin{aligned}
\hat{a}_k &= u_k \hat{A}_k + v_{-k}^* \hat{A}_{-k}^\dagger \\
\hat{a}_k^\dagger &= u_k^* \hat{A}_k^\dagger + v_{-k} \hat{A}_{-k}
\end{aligned}
\tag{4.6}
$$

This transformations introduces a new set of operators $\hat{A}_k$ and $\hat{A}_k^\dagger$ on which we impose the same fermionic anti-commutation relation

$$
\{\hat{A}_k, \hat{A}_{k'}^\dagger\} = \delta_{k,k'}
\tag{4.7}
$$

as obeyed by the original particle operators $\hat{a}_k$ and $\hat{a}_k^\dagger$, to make the transformation canonical.

It can be easily verified that Eq. (4.7) are satisfied if

$$
|u_k|^2 + |v_{-k}|^2 = 1 \quad \text{(for any k)}
\tag{4.8}
$$

Therefore u_k and v_{-k} can be chosen parametrically as

$$
u_k = \cos\theta_k \quad \text{and} \quad v_{-k} = \sin\theta_k
\tag{4.9}
$$

By inserting Eq (4.6) into Eq. (4.5) we obtain

$$\hat{\mathbf{H}} = E_{\mathbf{F}} + \frac{1}{2}\sum_k{}'(E_k + 2\rho g)2|v_{-k}|^2 + \frac{1}{2}\sum_k{}'(E_k + 2\rho g).(|u_k|^2 - |v_{-k}|^2).(\hat{A}_k^\dagger \hat{A}_k + \hat{A}_{-k}^\dagger \hat{A}_{-k})$$

$$+\frac{1}{2}\sum_k{}'(E_k + 2\rho g)2u_k v_{-k}(\hat{A}_k^\dagger \hat{A}_{-k}^\dagger + \hat{A}_k \hat{A}_{-k}) \qquad (4.10)$$

In order to make $\hat{\mathbf{H}}$ diagonal in $\hat{A}_k$ and $\hat{A}_k^\dagger$ we use the freedom to eliminate the last term of Eq. (4.10) i.e.,

$$\frac{1}{2}\sum_k{}'(E_k + 2\rho g)2u_k v_{-k} = 0 \qquad (4.11)$$

With Eq. (4.9) and by defining $\frac{1}{2}(E_k + 2\rho g)$ as α_k, Eq (4.11) gives for each k-th mode

$$\alpha_k \sin 2\theta_k = 0 \qquad (4.12)$$

The above condition is satisfied both for positive to negative values of α_k . The allowed values of the coefficient θ_k are

$$\theta_k = \pm\frac{m\pi}{2} \qquad (m\text{=integer}) \qquad (4.13)$$

It is easy to note that for the above θ_k values, the first term of Eq (4.10) corresponding to $\bar{E}$ we obtain

$$\bar{E} = E_{\mathbf{F}} + \frac{1}{2}\sum_k{}'(E_k + 2\rho g)2|v_{-k}|^2$$

$$= E_{\mathbf{F}} + \frac{1}{2}\sum_k{}'(E_k + 2\rho g)2\sin^2 2\theta_k \qquad (4.14)$$

Since $\frac{1}{2}\sum_k{}'(E_k + 2\rho g)2\sin^2 2\theta_k = 0$ we have $\bar{E} = E_{\mathbf{F}}$. This is the same result in the lowest order approximation obtained earlier. So the $\hat{\mathbf{H}}$ in Eq. (4.10) is modified as

$$\hat{\mathbf{H}} = E_{\mathbf{F}} + \frac{1}{2}\sum_k{}'(E_k + 2\rho g).(|u_k|^2 - |v_{-k}|^2).(\hat{A}_k^\dagger \hat{A}_k + \hat{A}_{-k}^\dagger \hat{A}_{-k}) \qquad (4.15)$$

Identifying $\frac{1}{2}(E_k + 2\rho g).(|u_k|^2 - |v_{-k}|^2) = \alpha_k \cos 2\theta_k = \alpha_k$, the Hamiltonian in its final form can be written as

$$\hat{\mathbf{H}} = E_{\mathbf{F}} + \frac{1}{2}\sum_k{}'(E_k + 2\rho g)(\hat{A}_k^\dagger \hat{A}_k + \hat{A}_{-k}^\dagger \hat{A}_{-k})$$

$$= E_{\mathbf{F}} + \frac{1}{2}\sum_k{}'\epsilon_k(\hat{A}_k^\dagger \hat{A}_k + \hat{A}_{-k}^\dagger \hat{A}_{-k}) \qquad (4.16)$$

where the quasi-particle energy ϵ_k is given by

$$\epsilon_k = \frac{\hbar^2 k^2}{2m} + 2\rho g = E_k + 2\rho g \qquad (4.17)$$

$E_k = \frac{\hbar^2 k^2}{2m}$ stands for the free particle energy if the interaction $g = 0$. The operator $\hat{A}_k^\dagger \hat{A}_k$ resembles a particle number operator and has eigenvalues 0 and 1. Hence the coherent state is determined by the requirement that

$$\hat{A}_k |\mathbf{y}(\theta)\rangle = 0 \quad \text{for all } k \neq \mathbf{F} \qquad (4.18)$$

Furthermore, all quasi-particle states correspond to different numbers of non-interacting fermions. The expression (4.16) is the fermionic counterpart of the Bogoliubov Hamiltonian for bosons. $E_\mathbf{F}$ term originating purely as kinetic energy term for bosonic case is zero since the coherent state corresponds to $k = 0$ (instead of $k = \mathbf{F}$). Secondly the ground state is the vacuum of the quasiparticle operators $\hat{A}_k$, $\hat{A}_k^\dagger$. In general, the spectrum is gapped [14, 30] everywhere corresponding to a non-zero difference between the first excited state and the ground state. The theory does not put any restriction upon the sign of g and hence it can be both positive and negative [11–14]. For a critical negative g i.e., g_c it is possible that ϵ_k becomes zero when $E_k = -2\rho g_c$. In this case the spectrum can be gapless [30]. This is in consistent with the general perception that in the case of spontaneously broken continuous symmetry, there is always a low-lying excitation energy ϵ_k which satisfies $\epsilon_k \longrightarrow 0$ as its momentum $k \longrightarrow 0$. In the present context quasi-particle energy ϵ_k smoothly goes to zero without a gap as the momentum goes to zero, even though the dispersion relation is quadratic in momentum rather than linear unlike the Bose gas.

The coherent state $|\mathbf{y}(\theta)\rangle$ is annihilated by all $\hat{A}_k$, $(k \neq \mathbf{F})$. The transformation (4.6) can be represented as

$$\hat{A}_k = \hat{\mathscr{U}}(\theta) \hat{a}_k \hat{\mathscr{U}}(\theta)^\dagger \qquad (4.19)$$

$$\hat{A}_k^\dagger = \hat{\mathscr{U}}(\theta) \hat{a}_k^\dagger \hat{\mathscr{U}}(\theta)^\dagger \qquad (4.20)$$

$\hat{\mathscr{U}}(\theta)$ represents the unitary displacement operator which produces the coherent state as given in Eq (3.13).

$$|\mathbf{y}(\theta)\rangle = \hat{\mathscr{U}}(\theta)|0\rangle \qquad (4.21)$$

The connection between $\hat{\mathscr{U}}(\theta)$ and $\hat{\mathbf{D}}(\mathbf{y})$ in Eq. (3.13) can be established as follows.

$$
\begin{aligned}
\hat{\mathscr{U}}(\theta)|0\rangle &= \hat{U}(\theta).\hat{\mathbf{D}}(\mathbf{y})|0\rangle \\
&= \hat{U}(\theta)|\mathbf{y}\rangle \\
&= \exp\left(i\theta\hat{N}\right)|\mathbf{y}\rangle \\
&= |e^{i\theta}\mathbf{y}\rangle \quad\quad\quad (4.22)
\end{aligned}
$$

Under the rotation of all Grassmann variables y_k by the same angle 'θ' i.e., the phase transformation of $y_k \to e^{i\theta}y_k$ [15] the displacement operator produces altogether a different coherent state $|e^{i\theta}\mathbf{y}\rangle$. But for a physical rotation "θ" is allowed to run from 0 to 2π. So, the permissible values of 'θ' for which Eqs (4.19) and (4.20) are valid are $\theta = 0, \pm\frac{\pi}{2}, \pm\pi, \pm\frac{3\pi}{2}$ as obtained from Eq. (4.13). These values of 'θ' are the same as obtained earlier by Braungardt *et. al.* in the context of Fermi gas in optical lattice in 1d [30]. For these specific choices of 'θ', the phase of the coherent state gets stabilized.

V.   CONCLUSION

In this article we have tried to understand the coherent state of a weakly interacting ultracold Fermi gas as a close analogue of Bose condensate. Notwithstanding statistical differences, the coherent state of fermions bear a close kinship with that for bosons. Since the fermionic operators anticommute their eigenvalues are anticommuting numbers. These numbers or Grassmann variables play an important part in the formulation of the present theoretical scheme. The fermionic coherent state is not invariant under rotation $e^{i\theta\hat{N}}$ with number operator $\hat{N}$ for fermions, while the Hamiltonian remains unitarily invariant. For *preferred choices of θ, the coherent and the rotated coherent state are therefore macroscopically distinct states because of spontaneous symmetry breaking.* The description of the coherent state of weakly interacting fermions and the rules of Grassmann algebra allow us to consider the appropriate thermodynamic limit of the system. Bogoliubov approximation can be implemented within the framework of this scheme to realize this coherent state as a state of macroscopic coherence of fermions. Unlike the bosonic case the lowest order approximation on the Hamiltonian describing the weakly interacting fermions yields an energy contribution due to kinetic energy of the particles only. The quasi-particle spectrum arising from the higher order interaction exhibits in general, the gaps [14, 30]. Our analysis reveals that

the close parallels between the phase space quasi-probabilities of the boson and fermions as pointed out by Cahill and Glauber [15] have deep rooted consequences in the physics of ultracold degenarate Fermi gas.

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